Contests with a Prize Externality and Stochastic Entry*

XiaoGang Che† Brad R. Humphreys‡§

Durham University West Virginia University

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Abstract

We analyze a contest with stochastic participation and a prize externality. A unique symmetric equilibrium exists in the contest. We demonstrate that the presence of a prize externality affects individual equilibrium spending but active participants always face the same expected payoff as in a contest without a prize externality. A positive prize externality gives a higher impact on individual equilibrium spending than a negative prize externality. Regardless of the existence and the sign of a prize externality, ex-post over-dissipation occurs if the actual number of participants is sufficiently large. Independent of the prize externality's sign, active participants spend less but face a higher payoff compared to a fixed-participation contest with the same expected number of players.

JEL classification: C72, D72, D82

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†Durham University Business School, Millhill Lane, Durham DH1 3LB, United Kingdom. Email: chexiaogang0925@gmail.com.

‡Department of Economics, West Virginia University, 1601 University Avenue, P.O. Box 625, Morgantown, WV 26506, United States. Email: brhumphreys@mail.wvu.edu.

§Corresponding author.

1 Introduction

The pioneering work on contests by Tullock (1980) spawned a vast body of literature on contest theory (Dixit, 1987; Baye and Shin, 1999; Szidarovszky and Okuguchi, 1997; Cornes and Hartley, 2005; Chowdhury and Sheremeta, 2011)[1]. The typical setup features participants competing to win a fixed prize in a contest where each participant decides on optimal effort, which, in turn, affects the probability of winning.[2] In the last two decades, contest theory has been applied in a variety of settings, including rent seeking (Hillman, 1989; Nitzan, 1994), R&D and patent races (Loury, 1979; Nti, 1997), political conflict (Hirshleifer, 1991; Garfinkel and Skaperdas, 2007), and the reward structure in labor markets (Rosen, 1986). Economists have also used contest theory to analyze issues in individual and team sports (Szymanski, 2003) and countries or cities competing for mega sporting events such as the World Cup or the Olympics (Corchon, 2000).

We analyze a contest where participation is stochastic and the final prize is endogenously determined by participants' spending or effort. In such a contest, each participant's effort will impact both the probability of winning and the size of the prize. We interpret this effect on the size of the prize as a prize externality, which can be either *positive* or *negative*. A positive (negative) prize externality generates an increase (decrease) in the size of the final prize when a participant invests more effort. Moreover, in the contest, each player only knows the total number of potential players and the independent probability of participation, rather than the actual number of players entering the contest. We are interested in identifying the impacts of a prize externality and stochastic entry on individual spending, expected payoff, total spending, and rent dissipation in the contest.

We first prove that there exists a unique symmetric pure-strategy equilibrium in the contest. We then show that individual equilibrium spending under a negative (positive) prize externality is strictly lower (greater) than it is in the contest where the prize is exogenously given. However, this does not hold for a player's expected payoff. Interestingly, participants will face the same expected payoff, no matter whether or not a prize externality exists in the contest. We refer to this property as *prize-externality independence on expected payoff*. We further show that a positive prize externality will have a larger impact on individual equilibrium spending than a negative prize externality. Moreover, this difference between

[1]See Corchon (2007), and Konrad (2007) for surveys of the contest theory literature.

[2]Note we use "expenditure", "effort", "spending", and "investment" interchangeably.

the two impacts increases as the level of the prize externality increases.

We also examine rent dissipation in the contest. It is already well known that when the number of participants is stochastic in a contest, ex-post over-dissipation may occur, in that the actual total spending or effort exceeds the final prize. Given the model setup in our paper, it is natural to examine whether or not the existence of a prize externality will affect the conditions under which ex-post over-dissipation occurs. Surprisingly, our finding shows that regardless of the existence, the sign and the level of a prize externality, once the actual number of participants is sufficiently large, i.e., greater than a threshold, the final prize will be over-dissipated in the contest. Finally, we compare the contest to a fixed-participation contest. In particular, we focus on the case where the expected number of players is the same in both contests. Regardless of the sign of the prize externality, individual equilibrium spending is always strictly lower, but an active participant obtains a higher payoff in the former contest than in the latter.

In a number of actual contests, total entry may be stochastic and unknown by players, and the size of the final prize may be significantly affected by participating players' expenditures. Our study provides a number of implications for these competitions. The implication related to a positive prize externality in the contest model is large-jackpot, long-odds lottery games like PowerBall, MegaMillions, EuroMillions, or Lotto 6/49 that are common throughout the world. In these games, any individual purchasing a lottery ticket cannot know how many others will purchase tickets, and the size of the jackpot, which depends on the number of tickets sold, is also unknown until the drawing. Our study provides an explanation for the link between lottery jackpot size and the increased purchase of lottery tickets noted by Cook and Clotfelter (1993), Farrell and Walker (1999), Matheson and Grote (2004) and others.

The analysis of a negative prize externality may help us understand the multilateral military conflict among several countries. In such a conflict, the spending level of a country on military defense determines the level of military power and, thus, the probability of winning a conflict; all other resources are devoted to productive activities, determining the size of aggregate output. Once the conflict begins, the victor collects the entire joint output of all countries as the prize. Obviously, given limited resources, investment in military spending lowers the production of other goods and services.[3] Each country may not know the exact number of other countries that may be drawn into the conflict and must make the optimal

[3]A similar tradeoff in conflicts has been studied by Hirshleifer (1991), Garfinkel and Skaperdas (2007), Hwang (2012) and others. However, they restrict their attention to the case of two countries.

choice of dividing its endowed resources between military spending and production. Also, a negative prize externality could occur in rent-seeking lobbying contests. As the expenditure on lobbying activities increases, the politician being lobbied might become concerned about public perception of the amount of largess showered on her by lobbyists and reduce the value of the prize awarded to the winner. Our results provide some insights about players' behavior in those contests.

In the existing contest theory literature, only a few studies analyze a prize externality. Chung (1996) analyzed a contest with a positive externality associated with players' expenditures and concluded that the contest always generates social waste. Shaffer (2006), considering both positive and negative prize externalities, showed that a player may obtain a higher payoff, compared to no prize externality. However, his model only examined the case of two players.[4] Chowdhury and Sheremeta (2011), and Baye, Kovenock, and de Vries (2012) characterized equilibria in a two-player contest with a prize externality, which they call a "spillover" effect. However, they did not include stochastic participation in the contest. Compared to previous research, we fully characterize the changes in individual spending and expected payoffs for participants when there exists a prize externality, and show the interesting property that a prize externality only affects individual equilibrium spending; an active participant always obtains the same expected payoff, no matter whether or not a prize externality exists in the contest.

Rent dissipation in contests where participation is stochastic has been studied in the literature. Higgins, Shughart, and Tollison (1985) considered a symmetric mixed-strategy zero-profit equilibrium for contest participants, and showed that rent is completely dissipated in expectation, but ex-post under-, over-, or exact-dissipation may occur, depending on actual number of participants in the contest.[5] Lim and Matros (2009) showed that ex-post over-dissipation may occur when the actual number of participants is much larger than the expected number of participants. We show that ex-post over-dissipation is independent of the existence of a prize externality and only depends on how many actual participants enter the contest.

[4]In a different line of research, a contest with an uncertain prize value was analyzed by Wärneryd (2003, 2008), and others. Unlike our model, they assume that while the number of players is fixed and the final prize has a certain value, some or all players in the contest are not fully informed, so they only know the prior distribution of the final prize.

[5]Corcoran (1984), Corcoran and Karels (1985), Michaels (1988), and Wolfgang and Chun-Lei (1994) further discuss over-dissipation in contests.

Munster (2006) studied risk-averse players in a contest with population uncertainty, concluding that equilibrium spending is lower under risk aversion if the expected fraction of contest players is small. In a similar setting, Lim and Matros (2009) showed that individual expenditure is single-peaked but total expenditure is monotonically increasing with the probability of participation. Myerson and Wärneryd (2006) compared a contest with stochastic participation to a contest with a fixed number of participants, each with the same number of expected players, and showed that under stochastic participation, an active participant exerts less effort but always has a higher payoff. Our paper shows that this result still remains valid, regardless of the sign of a prize externality.

2 The model

Consider a contest with $n \geq 3$ risk neutral and identical potential participants. Each participant enters the contest with an independent probability ρ, where $\rho \in (0,1)$. After entry, all active participants, without knowing the actual total number of participants, simultaneously determine nonnegative effort or spending to compete for a single prize with value V. The spending of participant i is x_i, where $x_i \in \mathbb{R}_+$. x_i is not refundable, and all participants incur a unit marginal cost of effort. The probability of participant i winning the contest is described by a contest success function (**CSF**, henceforth) P_i.

We define V and P_i as follows. Let $e(x_i) = \alpha x_i$ reflect the effect of spending by participant i on the value of the prize, where $\alpha \in (0,1)$ indicates the level of a prize externality. The value of the final prize V is determined by the spending of all active participants by

$$V(x_i; \mathbf{M}_{-i}) = \bar{A} \pm \alpha x_i \pm \sum_{j \in \mathbf{M}_{-i}} \alpha x_j \tag{1}$$

where $\bar{A} > 0$ is the basic prize, and $\mathbf{M}_{-i}$ identifies all participants in the contest other than i. The positive (negative) sign reflects the presence of a positive (negative) prize externality that generates an increase (a decrease) in the size of the prize, when participants invest more effort in the contest.

The **CSF** P_i for participant i is

$$P_i(x_i;\mathbf{M}_{-i}) = \begin{cases} \frac{1}{|\mathbf{M}_{-i}|+1}, & \text{if } x_i = 0 \text{ and } x_j = 0 \text{ for all } x_j \in \mathbf{M}_{-i} \\ \frac{x_i}{x_i + \sum_{j\in\mathbf{M}_{-i}} x_j}, & \text{if } x_i > 0 \text{ and } x_j > 0 \text{ for all } x_j \in \mathbf{M}_{-i} \end{cases}$$

where $|\mathbf{M}_{-i}|$ denotes the cardinality of $\mathbf{M}_{-i}$. Obviously, P_i satisfies all standard axioms described by Skaperdas (1996) as commonly used in the literature.[6]

We assume that at least one participant enters the contest, implying that $\sum_{k=1}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k} = 1$. In this setting, a participant strictly prefers to exert positive effort in any equilibrium, because there exists a positive probability $(1-\rho)^{n-1}$ that all of the other potential participants are inactive except player i, and thus, $\sum_{j\in\mathbf{M}_{-i}} x_j = 0$. In this case, participant i wins the prize with certainty by investing a strictly positive level of expenditure ($\epsilon > 0$). However, if participant i chooses to spend zero, the probability of winning the prize is $1/n$. Any positive spending ($\epsilon > 0$) induces a higher probability of winning and higher payoff for player i. Thus, zero spending cannot occur in equilibrium.

Let $\Gamma(\bar{A}, \rho, n, \alpha)$ denote the contest descried above. Then we present the timing of the contest game as follows: first, nature draws the number of actual participants from n (Each participant is selected with an independent probability ρ). Then, active participants simultaneously choose their level of spending without knowing the actual number of participants. Finally, payoffs are awarded.

Participant i's payoff. In contest $\Gamma(\bar{A}, \rho, n, \alpha)$, conditional on participation, the expected payoff of player i who chooses x_i is

$$U_i(x_i) = \sum_{\mathbf{M}_{-i}\in\mathbf{\Omega}} \rho^{|\mathbf{M}_{-i}|}(1-\rho)^{|\mathbf{N}\setminus\mathbf{M}_{-i}|} P_i(x_i;\mathbf{M}_{-i}) V(x_i;\mathbf{M}_{-i}) - x_i, \tag{2}$$

where $\mathbf{N}$ distinguishes the set of potential participants from the $n-1$ others and $\mathbf{\Omega}$ denotes the set of all possible subsets of $\mathbf{N}$. Because participants are identical, we omit subscripts and write the payoff function and **CSF** as $U(.)$ and $P(.)$, respectively, in the following analysis for convenience. Also, we let $\Gamma(\bar{A}, \rho, n, \alpha)^-$ ($\Gamma(\bar{A}, \rho, n, \alpha)^+$) denote the contest with a negative (positive) prize externality.

[6] Riis and Clark (1998) axiomatize CSFs when players are different in their contest-relevant individual characteristics. Rai and Sarin (2009) generalize these axiomatic foundations to the case where a player can exert effort in multiple dimensions.

In the contest, we focus on a symmetric equilibrium. Let x^* (x^{**}) denote the symmetric individual equilibrium spending in contest $\Gamma(\bar{A},\rho,n,\alpha)^-$ ($\Gamma(\bar{A},\rho,n,\alpha)^+$). The first order conditions for participant i are

in contest $\Gamma(\bar{A},\rho,n,\alpha)^-$,

$$\mathcal{C}\left[\frac{\sum_{j\in\mathbf{M}_{-i}}x^*}{(x^*+\sum_{j\in\mathbf{M}_{-i}}x^*)^2}\left(\bar{A}-\alpha x^*-\sum_{j\in\mathbf{M}_{-i}}\alpha x^*\right)-\frac{\alpha x^*}{x^*+\sum_{j\in\mathbf{M}_{-i}}x^*}\right]=1, \tag{3}$$

and in contest $\Gamma(\bar{A},\rho,n,\alpha)^+$,

$$\mathcal{C}\left[\frac{\sum_{j\in\mathbf{M}_{-i}}x^{**}}{(x^{**}+\sum_{j\in\mathbf{M}_{-i}}x^{**})^2}\left(\bar{A}+\alpha x^{**}+\sum_{j\in\mathbf{M}_{-i}}\alpha x^{**}\right)+\frac{\alpha x^{**}}{x^{**}+\sum_{j\in\mathbf{M}_{-i}}x^{**}}\right]=1, \tag{4}$$

where $\mathcal{C}=\sum_{\mathbf{M}_{-i}\in\mathbf{\Omega}}\rho^{|\mathbf{M}_{-i}|}(1-\rho)^{|\mathbf{N}\setminus\mathbf{M}_{-i}|}$.

The following result is then guaranteed.

Proposition 1. *In contest $\Gamma(\bar{A},\rho,n,\alpha)$, when the prize externality is negative (positive), there exists a unique symmetric pure-strategy equilibrium where each active participant's spending is x^* (x^{**}) given in Equation (3) (Equation (4)).*

Proof. Suppose that all active participants except i choose a common effort $x^c>0$, we then consider the best response ($x_i>0$) of player i.

If the prize externality is negative, the second order condition of $U(x_i;x^c)$ with respect to x_i shows that

$$\frac{\partial^2U(x_i;x^c)}{\partial x_i^2}=2\mathcal{C}\left[-\frac{\sum_{j\in\mathbf{M}_{-i}}x^c}{(x_i+\sum_{j\in\mathbf{M}_{-i}}x^c)^3}V(x_i;\mathbf{M}_{-i})-\frac{\alpha\sum_{j\in\mathbf{M}_{-i}}x^c}{(x_i+\sum_{j\in\mathbf{M}_{-i}}x^c)^2}\right]. \tag{5}$$

We have $\frac{\partial^2U(x_i;x^c)}{\partial x_i^2}<0$ for any $x_i>0$. This demonstrates that $U(x_i;x^c)$ is strictly concave for any $x_i>0$. Thus, given that the common effort of all other active players is $x^c=x^*>0$, from Equation (3), the best response for player i should be x^*, and $U(x^*;x^*)$ reaches its global maximum; this also guarantees uniqueness.

If the prize externality is positive, the second order condition of $U(x_i;x^c)$ with respect to

x_i shows that

$$\frac{\partial^2 U(x_i;x^c)}{\partial x_i^2} = 2\mathcal{C}\left[\frac{\alpha \sum_{j\in\mathbf{M}_{-i}} x^c}{(x_i+\sum_{j\in\mathbf{M}_{-i}} x^c)^2} - \frac{\sum_{j\in\mathbf{M}_{-i}} x^c}{(x_i+\sum_{j\in\mathbf{M}_{-i}} x^c)^3} V(x_i;\mathbf{M}_{-i})\right]. \tag{6}$$

Again, $\frac{\partial^2 U(x_i;x^c)}{\partial x_i^2} < 0$ and $U(x_i;x^c)$ is strictly concave for any $x_i > 0$. Therefore, following the same argument used to prove part (I.), given $x^c = x^{**} > 0$, the unique best response for player i is x^{**}. □

3 Individual and total equilibrium spending

After showing the existence and uniqueness of the symmetric pure-strategy equilibrium, in this section we examine the impact of a prize externality on individual spending and total spending in the contest. Let $\bar{n} \geq 2$ denote the expected number of players in $\Gamma(\bar{A},\rho,n,\alpha)$. Given that there are $\binom{n-1}{|\mathbf{M}_{-i}|}$ different ways to make a set that has $|\mathbf{M}_{-i}|$ players from the set $\mathbf{N}$, we know that

$$\bar{n} = \sum_{k=1}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}k = n\cdot\rho,$$

and

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\rho^k(1-\rho)^{n-1-k}(\frac{\bar{n}}{k+1})(\frac{k+1}{\bar{n}}) = \sum_{k=0}^{n-1}\binom{n}{k+1}\rho^{k+1}(1-\rho)^{n-1-k}(\frac{k+1}{\bar{n}}).$$

For analytical convenience, we make the following transformations. Define $t = k+1$, and then Equation (2) can be simplified as follows

$$\begin{aligned} U(x_i) &= \sum_{k=0}^{n-1}\binom{n-1}{k}\rho^k(1-\rho)^{n-1-k}\frac{x_i}{x_i+kx^c}\left(\bar{A}\pm\alpha x_i\pm k\alpha x^c\right)-x_i \\ &= \frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}t\frac{x_i}{x_i+(t-1)x^c}\left(\bar{A}\pm\alpha x_i\pm(t-1)\alpha x^c\right)-x_i. \end{aligned} \tag{7}$$

The first order conditions, Equations (3) and (4), can be rewritten as follows:

If the prize externality is negative,

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\rho^k(1-\rho)^{n-1-k}\left[\frac{1}{x^*}\frac{k}{(k+1)^2}\left(\bar{A}-(k+1)\alpha x^*\right)-\frac{\alpha}{k+1}\right]=1,$$
$$\frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\left[\frac{1}{x^*}\frac{t-1}{t}\left(\bar{A}-t\alpha x^*\right)-\alpha\right]=1.$$

Rearranging the equation yields

$$x^*=\frac{1}{(1+\alpha)\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}\bar{A}. \tag{8}$$

If the prize externality is positive,

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\rho^k(1-\rho)^{n-1-k}\left[\frac{1}{x^{**}}\frac{k}{(k+1)^2}\left(\bar{A}+(k+1)\alpha x^{**}\right)+\frac{\alpha}{k+1}\right]=1,$$
$$\frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\left[\frac{1}{x^{**}}\frac{t-1}{t}\left(\bar{A}+t\alpha x^{**}\right)+\alpha\right]=1.$$

Rearranging the equation yields

$$x^{**}=\frac{1}{(1-\alpha)\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}\bar{A}. \tag{9}$$

We now examine how individual equilibrium spending will change when α increases. From the expressions for individual equilibrium spending above, the following result can be easily established.

Proposition 2. *If the prize externality is negative (positive), individual equilibrium spending strictly decreases (increases) as the level of the prize externality increases.*

Proposition (2) characterizes the relationship between a prize externality and individual equilibrium spending. The impact of a prize externality on individual equilibrium spending depends on both the level of and the sign of the prize externality. The intuition is obvious; the higher α, the less (more) incentive for an active participant to invest and the lower (higher) individual equilibrium spending in contest $\Gamma(\bar{A},\rho,n,\alpha)^-$ $(\Gamma(\bar{A},\rho,n,\alpha)^+)$.

We further discuss the impact of a prize externality on total equilibrium spending. Let $T(x)$ denote total equilibrium spending. Clearly, $T(x)$ is given by $T(x) = \sum_{k=1}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} kx = \bar{n}x$, where x here represents individual equilibrium spending in the contest. We then have that

$$T(x^*) = \frac{1}{(1+\alpha)} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A},$$

$$T(x^{**}) = \frac{1}{(1-\alpha)} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A}.$$

Proposition (2) shows that individual equilibrium spending decreases (increases) in contest $\Gamma(\bar{A}, \rho, n, \alpha)^-$ $(\Gamma(\bar{A}, \rho, n, \alpha)^+)$, as the level of the prize externality α increases. Intuitively, the total equilibrium spending should have the same property. Also, from the equations above, it is straightforward to see that when α increases, $T(x^*)$ decreases and $T(x^{**})$ increase. Formally

Lemma 1. *If the prize externality is negative (positive), total equilibrium spending strictly decreases (increases) as the level of the prize externality increases.*

4 Prize externalities

In this subsection, we compare contests with and without a prize externality. In particular, we are interested in examining the changes in individual equilibrium spending and expected payoffs of active participants in contest $\Gamma(\bar{A}, \rho, n, \alpha)$, in contrast to the absence of a prize externality. Further, we will discuss which sign of a prize externality provides a higher impact on individual equilibrium spending in the contest.

4.1 Compared with no prize externality

When $e(x) = 0$, $\Gamma(\bar{A}, \rho, n, \alpha)$ is a contest with an exogenous prize and stochastic participation, which has been studied by Myerson and Wärneryd (2006), and Lim and Matros (2009). Let $\Gamma(\bar{A}, \rho, n, \alpha = 0)$ denote this type of contest. The expected payoff of player i in contest $\Gamma(\bar{A}, \rho, n, \alpha = 0)$, denoted by $U_p(x_i)$, is given by

$$U_p(x_i) = \frac{1}{\bar{n}} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} t \frac{x_i}{x_i + (t-1)x_j} \bar{A} - x_i. \tag{10}$$

Further, let $\breve{x}$ denote symmetric equilibrium spending of each active participant in contest $\Gamma(\bar{A}, \rho, n, \alpha = 0)$, and $\breve{x}$ is given by

$$\frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\left(\frac{1}{\breve{x}}\frac{t-1}{t}\bar{A}\right) = 1 \Longleftrightarrow \breve{x} = \frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}\bar{A}. \tag{11}$$

Consider an active participant's equilibrium spending and expected payoffs in contests $\Gamma(\bar{A}, \rho, n, \alpha)$ and $\Gamma(\bar{A}, \rho, n, \alpha = 0)$. The following result can be easily derived.

Proposition 3. *(I.), $x^* < \breve{x} < x^{**}$; (II.) $U(x^*) = U(x^{**}) = U_p(\breve{x})$.*

Proof. From Equations (8), (9), and (11), it is straightforward to see that part (I.) holds for any $\alpha \in (0, 1)$. Next, we prove part (II.). Given the equilibrium individual spending x^*, x^{**}, and $\breve{x}$, we have that the expected payoff for player i who has been selected in $\Gamma(\bar{A}, \rho, n, \alpha)^-$ is given by

$$\begin{aligned}
U(x^*) &= \frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\left(\bar{A} - t\alpha x^*\right) - x^* \\
&= \frac{1}{\bar{n}}\bar{A} - \frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}t\alpha x^* - x^* \\
&= \frac{1}{\bar{n}}\bar{A} - (1+\alpha)x^*.
\end{aligned}$$

This gives that $U(x^*) = \frac{1}{\bar{n}}\bar{A}[1 - \sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}]$.

In $\Gamma(\bar{A}, \rho, n, \alpha)^+$, we have that

$$\begin{aligned}
U(x^{**}) &= \frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\left(\bar{A} + t\alpha x^{**}\right) - x^{**} \\
&= \frac{1}{\bar{n}}\bar{A} + \frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}t\alpha x^{**} - x^{**} \\
&= \frac{1}{\bar{n}}\bar{A} - (1-\alpha)x^{**},
\end{aligned}$$

This shows that $U(x^{**}) = \frac{1}{\bar{n}}\bar{A}[1 - \sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}]$.

Following the same procedure, an active participant's expected payoff in contest $\Gamma(\bar{A}, \rho, n, e = 0)$ can be rewritten as follows: $U_p(\breve{x}) = \frac{1}{\bar{n}}\bar{A}[1 - \sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}]$. We can then conclude that $U(x^*) = U(x^{**}) = U_p(\breve{x})$. $\square$

Part (I) of Proposition (3) demonstrates that, compared with the case of no prize externality, the presence of a negative prize externality in a contest reduces individual equilibrium spending of active participants, while a positive prize externality increases individual equilibrium spending. This result is intuitive; when a contest has a negative (positive) prize externality, this will lead to a smaller (larger) final prize. Thus, an active participant has an incentive to spend less (more) in the contest.

However, this does not mean that a negative prize externality will generate a lower equilibrium payoff for an active participant, or a positive prize externality will provide a higher payoff. Part (II) of Proposition (3) shows that no matter whether or not a prize externality exists in the contest, an active participant always faces the same expected payoff. Moreover, this result holds independent of the sign and the level of a prize externality. As mentioned above, we refer to this as prize-externality independence on expected payoff. This property implies that if there exists a prize externality in the contest, in other words, when the final prize is endogenously determined by the effort levels of all participants, no matter whether the sign is negative or positive, an active participant will adjust his investment to ensure that his expected payoff will be exactly equal to that of a contest without a prize externality.

This provides some interesting implications. For example, even though the prize structures for parimutuel forms of gambling (e.g., large-jackpot lotto games, horse race betting) and fixed prize gambling (e.g., scratch-card gambling, small jackpot lottery games) are different, the expected payoff for a player would be indifferent when participating in both types of gambling. However, individual equilibrium spending from players is higher in the parimutuel forms of gambling, suggesting that gambling designers can obtain a higher revenue from such gambling forms. Alternatively, in a rent-seeking game, if the politician being lobbied reduces the size of the prize awarded when the lobbying effort becomes excessive (too many first class airline tickets on fact finding junkets, too many lavish dinners), lobbyists will be indifferent to the presence of this disincentive.

From part (I) of Proposition (3), the following corollary can be easily derived.

Corollary 1. The total equilibrium spending is less (greater) under a negative (positive) prize externality, compared with the case of no prize externality.

4.2 The impact on individual spending

From the results above, a prize externality will not affect the expected payoff for an active participant, but affects individual equilibrium spending. Given this, do negative and positive prize externalities affect individual equilibrium spending in the same way? We next show that a positive prize externality will induce a larger change in individual equilibrium spending than a negative prize externality. Moreover, it turns out that the higher α, the larger the difference of the impacts of the two prize externalities. We present these results as follows:

Proposition 4. *Given the same level of α, the impact of a negative prize externality on individual equilibrium spending is less than that of a positive prize externality. Moreover, this difference between the two impacts increases as α increases.*

Proof. We have

$$(\breve{x} - x^*) = \frac{\alpha}{\bar{n}(1+\alpha)} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A},$$

and

$$(x^{**} - \breve{x}) = \frac{\alpha}{\bar{n}(1-\alpha)} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A}.$$

Obviously, $(\breve{x} - x^*) < (x^{**} - \breve{x})$ for any $\alpha \in (0,1)$, indicating that a positive prize externality has a higher effect on individual spending than that of a negative prize externality.

Define $\tau = (x^{**} - \breve{x}) - (\breve{x} - x^*) = \frac{2\alpha^2}{\bar{n}(1-\alpha)(1+\alpha)} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A}$. Differentiating τ with respect to α yields

$$\begin{aligned}\frac{\partial \tau}{\partial \alpha} &= \left[\frac{4\alpha}{1-\alpha^2} + \frac{4\alpha^2}{(1-\alpha^2)^2}\right] \frac{1}{\bar{n}} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A} \\ &= \frac{4\alpha}{\bar{n}(1-\alpha^2)^2} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A} > 0\end{aligned}$$

This suggests that the difference between the two impacts of a negative prize externality and a positive prize externality on individual equilibrium spending becomes larger, when α increases. □

5 Stochastic participation

5.1 Rent dissipation

As mentioned above, rent dissipation as a long-standing question has been discussed in contest theory literature. In this subsection, we examine rent dissipation in contests $\Gamma(\bar{A}, \rho, n, \alpha)^-$ and $\Gamma(\bar{A}, \rho, n, \alpha)^+$. It is easy to check that ex-ante over-dissipation will never take place in both contests. However, when the participation process is stochastic, the actual number of participants may be greater than the expected number of participants in the contest, and this makes a possibility that the final prize may be over-dissipated. Our analysis here focuses on how the presence of a prize externality will affect ex-post over-dissipation in the contest. In particular, we discuss the conditions under which ex-post over-dissipation will occur.

Let $T'(\kappa, x) = \kappa x$ denote the actual total spending, where κ is the actual number of participants, and x here denotes individual equilibrium spending in the contest. Ex-post over-dissipation occurs if the actual total expending is greater than the prize. Then we have the following result.

Lemma 2. *Contests $\Gamma(\bar{A}, \rho, n, \alpha)^-$ and $\Gamma(\bar{A}, \rho, n, \alpha)^+$ share the same κ^* such that if $\kappa > \kappa^*$, ex-post over-dissipation will take place.*

Proof. If the prize externality is negative, the final prize minus the actual total spending is given by

$$\begin{aligned}\bar{A} - \kappa\alpha x^* - T'(\kappa, x^*) &= \bar{A} - \kappa(1+\alpha)x^* \\ &= \bar{A} - \kappa\frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}\bar{A}.\end{aligned} \tag{12}$$

If the prize externality is positive, the final prize minus the actual total spending is given by

$$\begin{aligned}\bar{A} + \kappa\alpha x^{**} - T'(\kappa, x^{**}) &= \bar{A} - \kappa(1-\alpha)x^{**} \\ &= \bar{A} - \kappa\frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\frac{t-1}{t}\bar{A}.\end{aligned} \tag{13}$$

Equations (12) and (13) imply that ex-post over dissipation takes place, regardless of the sign and the level of a prize externality. Moreover, there should exist a κ^* such that $\bar{A} - \kappa^* \alpha x^* - T'(\kappa^*, x^*) = \bar{A} + \kappa^* \alpha x^{**} - T'(\kappa^*, x^{**}) = 0$. When $\kappa > \kappa^*$, ex-post over-dissipation will take place. □

Next, we address whether or not ex-post over-dissipation depends on the existence of a prize externality in the contest. In the absence of a prize externality, the final prize minus the actual total spending is

$$\bar{A} - T'(\kappa, \breve{x}) = \bar{A} - \kappa \breve{x} = \bar{A} - \kappa \frac{1}{\bar{n}} \sum_{t=1}^{n} \binom{n}{t} \rho^t (1-\rho)^{n-t} \frac{t-1}{t} \bar{A}. \tag{14}$$

Interestingly, all the rent-dissipation expressions above – Equations (12), (13), (14) – are independent of α. Therefore, whether or not ex-post over-dissipation occurs does not depend on the existence, sign, or level of a prize externality. Moreover, when the actual number of participants is equal to κ^*, we should have that $\bar{A} - T'(\kappa^*, \breve{x}) = \bar{A} - \kappa^* \alpha x^* - T'(\kappa^*, x^*) = \bar{A} + \kappa^* \alpha x^{**} - T'(\kappa^*, x^{**}) = 0$. We can summarize the result as follows:

Proposition 5. *If $\kappa > \kappa^*$, ex-post over-dissipation will take place, regardless of the existence, the sign and the level of a prize externality.*

Proposition (5) demonstrates that no matter whether or not a prize externality exists in the contest, when participation is stochastic, ex-post over-dissipation becomes a natural feature of the contest; it occurs if the actual number of participants is sufficiently large.[7]

5.2 Identical expected participation

Next, we consider how individual equilibrium spending and the expected payoff of an active participant will change in contests with and without stochastic participation. In particular, we focus on the comparison between contest $\Gamma(\bar{A}, \rho, n, \alpha)$ and a contest where the number of participants is known to be $\bar{n}$ with certainty, denoted by $\Gamma(\bar{A}, \rho = 1, \bar{n}, \alpha)$. Recall that $\bar{n} = n \cdot \rho$.

Given identical participants, the expected payoff for participant i in contest $\Gamma(\bar{A}, \rho =$

[7] Substituting $\bar{n}$ into Equation (14) shows that $\bar{A} - T'(\bar{n}, \breve{x}) > 0$. We should therefore have $\kappa^* > \bar{n}$.

$1, \bar{n}, \alpha)$, denoted by $U_c(x_i)$, is given by

$$U_c(x_i) = \frac{x_i}{x_i + (\bar{n}-1)x_j}\Bigg(\bar{A} \pm \alpha x_i \pm (\bar{n}-1)\alpha x_j\Bigg) - x_i, \tag{15}$$

where the negative (positive) sign reflects a negative (positive) prize externality. Let $x^{'}$ $(x^{''})$ denote individual equilibrium spending in a contest with a negative (positive) externality. In the symmetric equilibrium, when the prize externality is negative, $x^{'}$ should be satisfied by

$$\frac{1}{\bar{n}}\left[\frac{1}{x^{'}}\frac{(\bar{n}-1)}{\bar{n}}\Bigg(\bar{A} - \bar{n}\alpha x^{'}\Bigg) - \alpha\right] = 1 \Longleftrightarrow x^{'} = \frac{\bar{A}(\bar{n}-1)}{\bar{n}^2(1-\alpha)} \tag{16}$$

when the prize externality is positive, $x^{''}$ should be satisfied by

$$\frac{1}{\bar{n}}\left[\frac{1}{x^{''}}\frac{(\bar{n}-1)}{\bar{n}}\Bigg(\bar{A} + \bar{n}\alpha x^{''}\Bigg) + \alpha\right] = 1 \Longleftrightarrow x^{''} = \frac{\bar{A}(\bar{n}-1)}{\bar{n}^2(1+\alpha)}. \tag{17}$$

Clearly, in equilibrium, the expected payoff for an active participant can be re-written as follows: $U_c(x^{'}) = \frac{1}{\bar{n}}\bar{A} - (1+\alpha)x^{'}$, if there exists a negative prize externality, and $U_c(x^{''}) = \frac{1}{\bar{n}}\bar{A} - (1-\alpha)x^{''}$ if there exists a positive prize externality. The following results can then be easily derived.

Proposition 6. *Comparing contests* $\Gamma(\bar{A}, \rho, n, \alpha)$ *and* $\Gamma(\bar{A}, \rho = 1, \bar{n}, \alpha)$ *shows*

(I.) *if the prize externality is negative,* $x^* < x^{'}$*, and* $U(x^*) > U_c(x^{'})$*;*

(II.) *if the prize externality is positive,* $x^{**} < x^{''}$*, and* $U(x^{**}) > U_c(x^{''})$*.*

Proof. First define

$$\tau(t; x) = \frac{t-1}{t}\Bigg(\bar{A} \pm t\alpha x\Bigg), \tag{18}$$

where the positive (negative) sign reflects the presence of a positive (negative) prize externality. Regardless of the sign of a prize externality, we have that $\frac{\partial^2 \tau(t;x)}{\partial t^2} < 0$. Given that $\tau(t; x)$ is concave in t, applying Jensen's inequality shows that

$$\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\tau(t; x) < \tau(\bar{n}; x). \tag{19}$$

Consider the following two cases:

(I.) If the prize externality is negative, Equations (8), (16), and (19) indicate that

$$\frac{1}{\bar{n}}\sum_{t=1}^{n}\binom{n}{t}\rho^t(1-\rho)^{n-t}\left[\frac{1}{x^*}\tau(t;x^*)-\alpha\right] < \frac{1}{\bar{n}}\left[\frac{1}{x^*}\tau(\bar{n};x^*)-\alpha\right].$$

Given that the left-hand side of the inequality is equal to 1 (Equation (8)), it follows that

$$\frac{1}{\bar{n}}\left[\frac{1}{x^*}\frac{(\bar{n}-1)}{\bar{n}}\left(\bar{A}-\bar{n}\alpha x^*\right)-\alpha\right] > 1. \tag{20}$$

Further, define $L(x^*,\bar{n}) = \frac{1}{\bar{n}}[\frac{1}{x^*}\tau(\bar{n};x^*)-\alpha]$. Obviously, $\frac{dL(x^*,\bar{n})}{dx^*} < 0$; $L(x^*,\bar{n})$ is strictly decreasing in x^*. Thus, to make Equation (16) hold, x^* should be less than $x^{'}$.

Individual equilibrium payoffs in $\Gamma(\bar{A},\rho,n,\alpha)^-$ and $\Gamma(\bar{A},\rho=1,\bar{n},\alpha)$ are $U(x^*) = \frac{1}{\bar{n}}\bar{A} - (1+\alpha)x^*$ and $U_c(x^{'}) = \frac{1}{\bar{n}}\bar{A}-(1+\alpha)x^{'}$, respectively. Thus, when the prize externality is negative, we have that $U(x^*)$ is strictly greater than $U_c(x^{'})$ as $x^* < x^{'}$.

(II.) If the prize externality is positive, given that Equations (9), (17) and (19) hold, we follow the same logic from Part (I.) and have

$$\frac{1}{\bar{n}}\left[\frac{1}{x^{**}}\frac{(\bar{n}-1)}{\bar{n}}\left(\bar{A}+\bar{n}\alpha x^{**}\right)+\alpha\right] > 1. \tag{21}$$

Define $L'(x^{**},\bar{n}) = \frac{1}{\bar{n}}[\frac{1}{x^{**}}\tau(\bar{n};x^{**})+\alpha]$. Again, $L'(x^{**},\bar{n})$ is decreasing in x^{**} and, thus, we should have $x^{**} < x^{''}$ to make Equation (17) hold.

Individual equilibrium payoffs are $U(x^{**}) = \frac{1}{\bar{n}}\bar{A}-(1-\alpha)x^{**}$ in $\Gamma(\bar{A},\rho,n,\alpha)^+$ and $U_c(x^{''}) = \frac{1}{\bar{n}}\bar{A}-(1-\alpha)x^{''}$ in $\Gamma(\bar{A},\rho=1,\bar{n},\alpha)$. Clearly, we have that $U(x^{**})$ is greater than $U_c(x^{''})$. $\square$

Myerson and Wärneryd (2006) and Lim and Matros (2009) showed that an active player always invests less but gains a higher expected payoff in a contest where the number of players is a random variable distributed with mean $\bar{n}$, compared to a contest with exactly $\bar{n}$ players. Our results are consistent with theirs, indicating the robustness of the result in the presence of a prize externality.

From Proposition (6), it is easy to derive the following corollary.

Corollary 2. Total equilibrium spending is smaller in contest $\Gamma(\bar{A},\rho,n,\alpha)$ than in $\Gamma(\bar{A},\rho=1,\bar{n},\alpha)$, regardless of the sign of a prize externality.

6 Concluding remarks

We analyze a contest with participation uncertainty and a prize externality. In the contest, active participants do not know actual number of participants and make their decisions on spending or effort to compete for a single prize. The value of the final prize depends on total spending by all of active participants. This type of contest captures important features of a number of real-world competitions, like parimutuel forms of gambling, multilateral military conflicts among countries, and some rent-seeking contests. Our analysis provides some interesting insights into such contests.

A unique pure-strategy equilibrium exists in this contest. A negative prize externality lowers individual equilibrium spending, compared with an exogenously determined prize. If however, the prize externality is positive, an active participant spends more in the contest, i.e., individual equilibrium spending increases. Although the presence of a prize externality impacts individual equilibrium spending, the expected payoff of an active participant is identical to the outcome in the absence of a prize externality. This property implies that according to the level of a prize externality, an active player will strategically adjust his spending to earn the same expected payoff in the contest. We further show that the impact of a positive prize externality on individual equilibrium spending is greater than that of a negative prize externality. Moreover, the difference of the two impacts becomes larger, when the level of the prize externality increases.

Our analysis also studied rent dissipation in the contest. Ex-ante over-dissipation never occurs in the contest. However, given that the number of participants is stochastic, ex-post over-dissipation will take place when the actual number of participants is greater than a threshold. Interestingly, this feature does not depend on the existence, sign, or level of a prize externality. Finally, we compare individual equilibrium spending and expected payoffs in contests with and without participation uncertainty. Specifically, this comparison restricts attention to the same expected number of players. Although participation uncertainty reduces the level of individual equilibrium spending, a higher payoff for an active player is always guaranteed. This result holds independent of the sign of a prize externality,

Some clear extensions of our analysis exist. In the present paper, we consider only the case where the contribution of an active participant's spending on the value of the final prize is linear. However, the functional relationship between the two variables normally follows the law of diminishing returns, and therefore it would be interesting to examine whether or not our results still hold when the prize externality function is increasing-concave. Another interesting extension would be to examine the impacts of a prize externality under dynamic contest settings, where the presence of a prize externality may not only affect an active player's expenditure decisions in the current stage, but also the following stages.

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